

GAUSSIAN RULES FOR MÜNTZ SYSTEMS IN THE BOUNDARY ELEMENT METHOD

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Abstract: Generalized Gaussian quadrature rules for Müntz systems are considered. In a particular but very important case, the n -point quadrature formula which integrates exactly functions of the form $f(x) = f_1(x) + f_2(x)\log x$, where $f_1(x)$ and $f_2(x)$ are arbitrary algebraic polynomials of degree at most $n-1$, is obtained. In order to illustrate the efficiency of the obtained formula a numerical example is presented.

1. Introduction

In many computational applications in engineering, the boundary element method (BEM) requires the numerical evaluation of integrals with an end-point logarithmic singularity (cf. Katsikadelis [1]). In this short note we provide a numerical quadrature method for such a kind of integrals. These Gaussian quadrature rules possess several properties of the classical Gaussian formulae (for polynomial systems) and they can be applied to the wide class of functions, including smooth functions, as well as functions with end-point singularities, such as those appeared in the boundary-contact value problems, integral equations, etc.

2. Gaussian Rules for Müntz Systems

Gaussian integration can be extended in a natural way to non-polynomial functions, taking a system of linearly independent functions

$$\{P_0(x), P_1(x), P_2(x), \dots\} \quad (x \in [a, b]), \quad (1)$$

usually chosen to be complete in some suitable space of functions. If $w(x)$ is a given nonnegative weight on $[a, b]$ and the quadrature rule

$$\int_a^b f(x)w(x) dx = \sum_{k=1}^n A_k f(x_k) + R_n(f) \quad (2)$$

is such that it integrates exactly the first $2n$ functions in (1), we call this rule as *generalized Gaussian quadrature with respect to the system* (1). The existence and uniqueness of such a generalized Gaussian quadrature rule is always guaranteed if the first $2n$ functions of this system constitute a Chebyshev system on $[a, b]$. Then, all the weights A_1, \dots, A_n in (2) are positive. The generalized Gaussian quadratures for Müntz systems goes back to Stieltjes in 1884. Taking $P_k(x) = x^{\lambda_k}$ on $[a, b] = [0, 1]$, where $0 \leq \lambda_0 < \lambda_1 < \dots$, Stieltjes showed the existence of Gaussian formulae. In 1996 Ma, Rokhlin and Wandzura [3] gave a numerical algorithm for constructing the generalized Gaussian quadratures, but their algorithm is ill conditioned (see [3]). Recently Milovanović and Cvetković [5] presented an alternatively numerical method for constructing the generalized Gaussian quadrature (2) for Müntz polynomials, which is exact for each $f \in M_{2n-1}(\Lambda) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_{2n-1}}\}$.

This method is rather stable and simpler than the previous one, since it is based on construction and stable computation of orthogonal Müntz systems, previously developed in [4]. The method performs calculations in double precision arithmetics in order to get double precision results.

An interesting case of these rules can be obtained if we take $\lambda_{2k} = \lambda_{2k+1} = k$, $k \in \mathbb{N}_0$. Then our system becomes $M_{2n-1}(\Lambda) = \text{span}\{1, \log x, x, x \log x, \dots, x^{n-1}, x^{n-1} \log x\}$ and the corresponding n -point Gaussian quadrature formula (2) is exact for all functions of the form $f(x) = f_1(x) + f_2(x) \log x$, where $f_1(x)$ and $f_2(x)$ are algebraic polynomials of degree at most $n-1$. The parameters of such quadratures for a given n can be obtained by using algorithms from [6]. For example, for $n=5$, $n=10$, and $n=15$, we have the following lists of nodes and weights,

```
{node5,weight5}={{0.00565222820508010,0.0734303717426523,0.284957404462558,
0.619482264084778,0.915758083004698},{0.0210469457918546,
0.130705540744447,0.289702301671314,0.350220370120399,0.208324841671986}}
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{node10,weight10}={{0.000482961710689632,0.00698862921431577,
0.0326113965946776,0.0928257573891660,0.198327256895404,
0.348880142979353,0.530440555787956,0.716764648511655,
0.875234557506234,0.975245698684393},{0.00183340007378985,
0.0134531223459918,0.0404971943169584,0.0818223696589037,
0.129192342770138,0.169545319547259,0.189100216532996,
0.177965753961471,0.133724770615462,0.0628655101770317}}
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{node15,weight15}={{0.000105784548458628,0.00156624383616781,
0.00759521890320708,0.0228310673939862,0.0523886301568200,
0.100758685201213,0.170740768849943,0.262591206118993,0.373536505184558,
0.497746358414533,0.626789031392374,0.750516103461408,0.858255335207861,
0.940141291212346,0.988401595986342},{0.000403217724648462,
0.00306297843478701,0.00978421211876610,0.0215587522255813,
0.0383230673708892,0.0588981990263005,0.0811170299392595,
0.102122101972069,0.118789059030401,0.128210316446694,0.128163327417093,
0.117489465888492,0.0963230185695906,0.0661345398318932,
0.0296207140035362}}
```

Example 1. We apply the quadrature formula (2) to the integral

$$I = \int_0^1 (3 + \log x) \sin(\pi x) dx,$$

which value can be expressed as $(\text{Ci}(\pi) + 6 - \gamma - \log \pi) / \pi = 1.38519624952742\dots$. The relative errors in the corresponding Gaussian quadratures for $n=5$, 10 , and 15 are 1.12×10^{-4} , 4.30×10^{-12} , and 6.66×10^{-16} , respectively. On the other side in this example, a sequence of Gauss-Legendre quadratures converges much slower. For example, relative errors are 4.73×10^{-5} , 3.23×10^{-6} , and 5.58×10^{-9} for $n=10$, 20 , and 100 , respectively.

Some transformation methods for integrals with Müntz polynomials can be also found in [2].

3. References

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